ALGEBRAIC CYCLES ON SEVERI-BRAUER SCHEMES OF PRIME DEGREE OVER A CURVE

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Abstract. Let k be a perfect field and let p be a prime number different from the characteristic of k. Let C be a smooth, projective and geometrically integral k-curve and let X be a Severi-Brauer C-scheme of relative dimension $p-1$. In this paper we show that $CH^d(X)_{\text{tors}}$ contains a subgroup isomorphic to $CH_0(X/C)$ for every d in the range $2 \leq d \leq p$. We deduce that, if k is a number field, then $CH^d(X)$ is finitely generated for every d in the indicated range.

1. Introduction.

Let k be a perfect field with algebraic closure $\overline{k}$. Very little is known about algebraic cycles on algebraic k-varieties, especially in codimension greater than 2 or dimension greater than zero. Let p be a prime number different from the characteristic of k and let C be a smooth, projective and geometrically integral k-curve. In this paper we study a certain subgroup of $CH^d(X)_{\text{tors}}$ for a Severi-Brauer C-scheme $q\colon X \to C$ of relative dimension $p-1$ and any integer d such that $2 \leq d \leq p$. Let

$$CH_0(X/C) = \operatorname{Ker}\Big[CH_0(X) \xrightarrow{q_*} CH_0(C)\Big]$$

and let $\pi^*\colon CH^d(X) \to CH^d\big(\overline{X}\big)$ be induced by the extension-of-scalars map $\overline{X} \to X$, where $\overline{X} = X \otimes_k \overline{k}$. Then the following holds.

Main Theorem. *For any d as above, there exists a canonical isomorphism*

$$\operatorname{Ker}\Big[CH^d(X) \xrightarrow{\pi^*} CH^d\big(\overline{X}\big)\Big] \simeq CH_0(X/C).$$

Consequently, if k is a number field, then $CH^d(X)$ is finitely generated.

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2. Preliminaries.

Let k be a perfect field, fix an algebraic closure $\overline{k}$ of k and let $\Gamma = \mathrm{Gal}\big(\overline{k}/k\big)$. Now let C be a smooth, projective and geometrically integral k-curve and let X be a Severi-Brauer scheme over C [4, §8]. There exists a proper and flat k-morphism $q\colon X \to C$ all of whose fibers are Severi-Brauer varieties of dimension $m-1$ ($m \geq 1$) over the appropriate residue field [loc.cit.]. We will write X_η for the generic fiber $X \times_C \mathrm{Spec}\, k(C)$ of q and A for the central simple $k(C)$-algebra associated to X_η. We define

$$CH_0(X/C) = \mathrm{Ker}\Big[CH_0(X) \xrightarrow{q_*} CH_0(C)\Big].$$

Now let C_0 be the set of closed points of C. The group of *divisorial norms* of X/C (cf. [6]) is the group

$$k(C)^*_{\mathrm{dn}} = \{f \in k(C)^*\colon \forall y \in C_0, \mathrm{ord}_y(f) \in (q_y)_*(CH_0(X_y))\}$$

where, for each $y \in C_0$, $q_y\colon X_y \to \mathrm{Spec}\, k(y)$ is the structural morphism of the fiber X_y. This group is closely related to $CH_0(X/C)$ (see [2, Proposition 3.1]). Indeed, there exists a canonical isomorphism

$$CH_0(X/C) \simeq k(C)^*_{\mathrm{dn}}/k^*\mathrm{Nrd}\, A^*.$$

Now fix an integer d such that $1 \leq d \leq m$ and let

$$CH^d(X)' = \mathrm{Ker}\Big[CH^d(X) \xrightarrow{\pi^*} CH^d\big(\overline{X}\big)^{\Gamma}\Big],$$

where $\pi\colon \overline{X} \to X$ is the canonical map. A simple transfer argument shows that $CH^d(X)'$ is a subgroup of $CH^d(X)_{\mathrm{tors}}$. Now, since $\overline{X} \to \overline{C}$ has a section, $\overline{X}$ is a projective bundle over $\overline{C}$. Thus there exists an isomorphism

$$CH^d\big(\overline{X}\big) \simeq \mathbb{Z} \oplus CH_0\big(\overline{C}\big).$$

(see [3, Theorem 3.3(b), p.34]). Therefore, if $J_C(k)$ is finitely generated, where J_C is the Jacobian variety of C (e.g., k is a number field or $C = \mathbb{P}^1_k$), then $CH^d(X)$ is finitely generated if and only if $CH^d(X)'$ is finite.

3. The general method.

Let C be as above and let X be any smooth, projective and geometrically integral k-variety such that there exists a proper and flat morphism $q\colon X \to C$ whose generic fiber X_η is geometrically integral. We have an exact sequence [7]

(1)

$$H^{d-1}(X_\eta, \mathcal{K}_d) \xrightarrow{\delta} \bigoplus_{y\in C_0} CH^{d-1}(X_y) \to CH^d(X) \xrightarrow{j^*} CH^d(X_\eta) \to 0,$$

where $j\colon X_\eta \to X$ is the natural map and the map which we have labeled δ will play a role later when $k = \overline{k}$. A similar exact sequence exists over $\overline{k}$, and we have two natural exact commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} j^* \arrow[r] \arrow[d] & CH^d(X) \arrow[r] \arrow[d] & CH^d(X_\eta) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\operatorname{Ker} \overline{\jmath}^*)^{\Gamma} \arrow[r] & CH^d\big(\overline{X}\big)^{\Gamma} \arrow[r] & CH^d\big(\overline{X}_{\overline{\eta}}\big)^{\Gamma} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \frac{H^{d-1}(X_\eta, \mathcal{K}_d)}{j^* H^{d-1}(X, \mathcal{K}_d)} \arrow[r] \arrow[d] & \bigoplus_{y\in C_0} CH^{d-1}(X_y) \arrow[r, two heads] \arrow[d, "\oplus_{\overline{y}|y}\pi_{\overline{y}}^*"] & \operatorname{Ker} j^* \arrow[d] \\
0 \arrow[r] & \left(\frac{H^{d-1}\big(\overline{X}_{\overline{\eta}}, \mathcal{K}_d\big)}{\overline{\jmath}^* H^{d-1}\big(\overline{X}, \mathcal{K}_d\big)}\right)^{\Gamma} \arrow[r, "\overline{\delta}"] & \bigoplus_{\overline{y}|y} CH^{d-1}\big(\overline{X}_{\overline{y}}\big)^{\Gamma_y} \arrow[r] & (\operatorname{Ker} \overline{\jmath}^*)^{\Gamma}
\end{tikzcd} \tag{2}$$

where, for each $y \in C_0$, we have fixed a closed point $\overline{y}$ of $\overline{C}$ lying above y and written $\Gamma_y = \operatorname{Gal}\big(\overline{k}/k(y)\big)$. Set

$$CH^d(X_\eta)' = \operatorname{Ker}\Big[CH^d(X_\eta) \to CH^d\big(\overline{X}_{\overline{\eta}}\big)^{\Gamma}\Big]$$

and, for each $y \in C_0$,

$$CH^{d-1}(X_y)' = \operatorname{Ker}\Big[CH^{d-1}(X_y) \xrightarrow{\pi_{\overline{y}}^*} CH^{d-1}\big(\overline{X}_{\overline{y}}\big)^{\Gamma_y}\Big].$$

Now define

$$E\big(\overline{X}/\overline{C}\,\big) = \operatorname{Coker}\left[\frac{H^{d-1}(X_\eta, \mathcal{K}_d)}{j^* H^{d-1}(X, \mathcal{K}_d)} \longrightarrow \left(\frac{H^{d-1}\big(\overline{X}_{\overline{\eta}}, \mathcal{K}_d\big)}{\overline{\jmath}^* H^{d-1}\big(\overline{X}, \mathcal{K}_d\big)}\right)^{\Gamma}\,\right]. \tag{3}$$

Then, applying the snake lemma to the preceding diagrams, we obtain[1]

Proposition 3.1. *There exists a natural exact sequence*

$$\begin{aligned}\bigoplus_{y\in C_0} CH^{d-1}(X_y)' &\to \operatorname{Ker}\big[\, CH^{d}(X)' \to CH^{d}(X_\eta)'\big]\\ &\to \operatorname{Ker}\left[E\big(\overline{X}/\overline{C}\,\big) \to \bigoplus_{y\in C_0}\frac{CH^{d-1}\big(\overline{X}_{\overline{y}}\big)^{\Gamma_y}}{\pi_{\overline{y}}^{*}\, CH^{d-1}(X_y)}\right] \to 0\,,\end{aligned}$$

where $E\big(\overline{X}/\overline{C}\,\big)$ *is the group (3).*

As regards the right-hand group in the exact sequence of the proposition, the following holds. Let

$$H^{d-1}(X_\eta,\mathcal{K}_d)' = \operatorname{Im}\left[H^{d-1}(X_\eta,\mathcal{K}_d)\to H^{d-1}\big(\overline{X}_{\overline{\eta}},\mathcal{K}_d\big)^{\Gamma}\,\right]$$

and

$$\operatorname{Sal}_d(X/C) = \left\{ f\in H^{d-1}\big(\overline{X}_{\overline{\eta}},\mathcal{K}_d\big)^{\Gamma} : \forall y\in C_0,\ \overline{\delta}_{\overline{y}}(f)\in \pi_{\overline{y}}^{*}\, CH^{d-1}(X_y)\right\},$$

where $\overline{\delta}$ and $\pi_{\overline{y}}^{*}$ are the maps of diagram (2).

Proposition 3.2. *There exists a natural exact sequence*

$$\begin{aligned} 0 &\to \frac{\operatorname{Sal}_d(X/C)}{\big(\overline{\jmath}^{*}H^{d-1}\big(\overline{X},\mathcal{K}_d\big)\big)^{\Gamma}\cdot H^{d-1}(X_\eta,\mathcal{K}_d)'}\\ &\to \operatorname{Ker}\left[E\big(\overline{X}/\overline{C}\,\big) \to \bigoplus_{y\in C_0}\frac{CH^{d-1}\big(\overline{X}_{\overline{y}}\big)^{\Gamma_y}}{\pi_{\overline{y}}^{*}\, CH^{d-1}(X_y)}\right]\\ &\to H^{1}\big(\Gamma,\overline{\jmath}^{*}H^{d-1}\big(\overline{X},\mathcal{K}_d\big)\big)\,.\end{aligned}$$

Proof. This follows by applying the snake lemma to a diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r]\arrow[d] & B \arrow[r]\arrow[d] & B/A \arrow[r]\arrow[d] & 0 \\ 0 \arrow[r] & \overline{A}^{\Gamma} \arrow[r] & \overline{B}^{\Gamma} \arrow[r] & \big(\overline{A}/\overline{B}\big)^{\Gamma} \arrow[r] & H^1\big(\Gamma,\overline{A}\big) \end{tikzcd}$$

with $\overline{A}=\overline{\jmath}^{*}H^{d-1}\big(\overline{X},\mathcal{K}_d\big)$, $\overline{B}=H^{d-1}\big(\overline{X}_{\overline{\eta}},\mathcal{K}_d\big)$, etc. □

[1] Proposition 3.1 was inspired by [1, Proposition 1.1].

4. Proof of the main theorem.

Let C and A be as in Section 2, let p be a prime number different from the characteristic of k and let X be a Severi-Brauer scheme over C of relative dimension $p-1$.

Lemma 4.1. *There exists a Γ-isomorphism*

$$\overline{\jmath}^* H^{d-1}\big(\overline{X}, \mathcal{K}_d\big) \simeq \overline{k}^*.$$

Proof. Clearly, $\overline{\jmath}^* H^{d-1}\big(\overline{X}, \mathcal{K}_d\big)$ is the kernel of the map

$$\overline{\delta}\colon H^{d-1}\big(\overline{X}_{\overline{\eta}}, \mathcal{K}_d\big) \to \bigoplus_{\overline{y}|y} CH^{d-1}\big(\overline{X}_{\overline{y}}\big)$$

appearing in the exact sequence (1) over $\overline{k}$. Now $\overline{X}_{\overline{\eta}} \simeq \mathbb{P}^{p-1}_{\overline{\eta}}$ and $\overline{X}_{\overline{y}} \simeq \mathbb{P}^{p-1}_{\overline{k}}$ for every $\overline{y}$, whence we have Γ-isomorphisms

$$H^{d-1}\big(\overline{X}_{\overline{\eta}}, \mathcal{K}_d\big) \simeq \overline{k}(C)^*$$

and

$$CH^{d-1}\big(\overline{X}_{\overline{y}}\big) \simeq \mathbb{Z}$$

for each $\overline{y}$. Under these isomorphisms, the map $\overline{\delta}$ above corresponds to the canonical map

$$\overline{k}(C)^* \to \bigoplus_{\overline{y}|y} \mathbb{Z},$$

$$f \mapsto (\operatorname{ord}_{\overline{y}}(f))_{\overline{y}|y},$$

which yields the lemma. □

Theorem 4.2. *For every d such that $2 \le d \le p$, there exists a canonical isomorphism*

$$CH^d(X)' \simeq CH_0(X/C).$$

Proof. By Lemma 4.1, Hilbert's Theorem 90 and Proposition 3.2, there exists a natural isomorphism

$$\operatorname{Ker}\left[E\big(\overline{X}/\overline{C}\,\big) \to \bigoplus_{y\in C_0} \frac{CH^{d-1}\big(\overline{X}_{\overline{y}}\big)^{\Gamma_y}}{\pi^*_{\overline{y}}\, CH^{d-1}(X_y)}\right] \simeq \frac{\operatorname{Sal}_d(X/C)}{k^* H^{d-1}(X_\eta, \mathcal{K}_d)'}.$$

On the other hand, by [5,(8.7.2)], $H^{d-1}(X_\eta, \mathcal{K}_d)' = \operatorname{Nrd} A^*$ for every d such that $2 \le d \le p$ and, for each $y \in C_0$,

$$\pi^*_{\overline{y}}\, CH^{d-1}(X_y) \simeq \pi^*_{\overline{y}}\, CH^{p-1}(X_y) \simeq (q_y)_* CH_0(X_y) \quad (= \mathbb{Z} \text{ or } p\mathbb{Z}).$$

The latter implies that $\operatorname{Sal}_d(X/C) = k(C)^*_{\mathrm{dn}}$, whence

$$\begin{aligned}\operatorname{Sal}_d(X/C)/k^* H^{d-1}(X_\eta, \mathcal{K}_d)' &\simeq k(C)^*_{\mathrm{dn}}/k^* \operatorname{Nrd} A^* \\ &\simeq CH_0(X/C).\end{aligned}$$

Finally, [loc.cit.] shows that the groups $CH^d(X_\eta)$ and $CH^{d-1}(X_y)$ ($y \in C_0$) are torsion free, whence $CH^d(X_\eta)'$ and $CH^{d-1}(X_y)'$ vanish. The theorem now follows from Proposition 3.1. □

Corollary 4.3. *Let d be such that $2 \le d \le p$. Then $CH^d(X)'$ is finite if*

(1) *k is a number field, or*
(2) *k is a field of finite type over $\mathbb{Q}$, $C = \mathbb{P}^1_k$ and X has a 0-cycle of degree one.*

In each of these cases, the group $CH^d(X)$ is finitely generated.

Proof. Indeed, in these cases the group $CH_0(X/C)$ is finite [2]. □

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